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# THE AXES OF A QUADRATIC VECTOR.

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## 1. Introduction.

The well-known *linear vector function* [1] permits of several kinds of extension or generalization. We may, for example, have a function linear in each of two vectors, as $\varphi(\rho, \sigma)$; or we may have $F[\rho]$ quadratic in a single vector $\rho$. These two concepts evidently merge into one another, for $\varphi(\rho, \sigma)$ becomes a quadratic function of $\rho$ when we let $\sigma$ equal $\rho$.

In a former paper [2] a study was made of various types of vectors quadratic in a single vector $\rho$. The following results will be fundamental to the present discussion:

(A) An axis of $F\rho$ is a direction $\beta$ such that $F\beta$ is parallel to $\beta$ or else null. A quadratic vector has in general seven axes. If six axes lie on a quadric cone there are an infinite number of axes, and conversely; the vector function is then *reducible.* The number of *distinct* axes may be less than seven, since an axis may be of multiple order.

(B) In general any quadratic vector may be written as the sum of two terms

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[1] Hamilton, Elements of Quaternions, Chap. II, Section 6; Gibbs-Wilson, Vector Analysis, Chap. V.

[2] Proc. Amer. Acad. Arts & Sci., **52**, No. 7 (Jan. 1917), pp. 369–454.

$$F\rho = V\varphi\rho\theta\rho + \rho S\delta\rho \tag{1}$$

where $\varphi$ and $\theta$ are linear vector functions and $\delta$ is a vector. Exception can only occur when one of the axes is of order at least five.

(C) The term $V\varphi\rho\theta\rho$ may in general be expanded as

$$V\varphi\rho\theta\rho = \alpha_1 x_2 x_3 + \alpha_2 x_3 x_1 + \alpha_3 x_1 x_2 \tag{2}$$

where $\alpha_1$, $\alpha_2$, and $\alpha_3$ are constant vectors, and where the $x$'s are given by

$$x_1 S\beta_1\beta_2\beta_3 = S\beta_2\beta_3\rho, \quad x_2 S\beta_1\beta_2\beta_3 = S\beta_3\beta_1\rho, \quad x_3 S\beta_1\beta_2\beta_3 = S\beta_1\beta_2\rho \tag{3}$$

and where the $\alpha$'s depend on nine scalars by the scheme

$$\begin{aligned} \alpha_1 &= A_{11}\beta_1 + A_{21}\beta_2 + A_{31}\beta_3 \\ \alpha_2 &= A_{12}\beta_1 + A_{22}\beta_2 + A_{32}\beta_3 \\ \alpha_3 &= A_{13}\beta_1 + A_{23}\beta_2 + A_{33}\beta_3 \end{aligned} \tag{4}$$

(D) In general the quadratic vector is fully determined when its axes are given, aside from the term $\rho S\delta\rho$ and a multiplicative scalar.

I shall refer to this former paper as C. Q. V.

## 2. The A's as Functions of the Axes.

It follows from ($C$) and ($D$) that the nine $A$'s are determinate as functions of the axes, aside from a common scalar multiplier. A knowledge of these functions facilitates our attack on a variety of problems. I propose to express these $A$'s in terms of the axes $\beta_1$, $\beta_2, \cdots \beta_7$ and to illustrate the utility of the results by some applications.

With the notation and results of pp. 377–384 of C. Q. V. we may write, omitting mere constant multipliers.

$$V\varphi\rho\theta\rho = \beta_4 \cdot \frac{(567)\,(P_5P_6P\rho)}{(P_5P_6P_7)} + \beta_5 \cdot \frac{(647)\,(P_6P_4P\rho)}{(P_6P_4P_7)} + \beta_6 \cdot \frac{(457)\,(P_4P_5P\rho)}{(P_4P_5P_7)} \tag{5}$$

Now it is evident from (2) that if we let $x_1 = 0$, $x_2 = x_3 = 1$, we shall reduce $V\varphi\rho\theta\rho$ to $\alpha_1$; but by p. 381 of C. Q. V. we shall then reduce $P\rho$ to $i$. We thus obtain

$$\alpha_1 = \beta_4 \cdot \frac{(567)\,(P_5P_6i)}{(P_5P_6P_7)} + \beta_5 \cdot \frac{(647)\,(P_6P_4i)}{(P_6P_4P_7)} + \beta_6 \cdot \frac{(457)\,(P_4P_5i)}{(P_4P_5P_7)} \tag{6}$$

The factor $(P_4P_5i)$ is one of the minors of the determinant (22) p. 381 of C. Q. V., which, developed by the method there given, yields

$$(P_4P_5i) = (123)\,(234)\,(235)\,(415) \tag{7}$$

Expanding the factors $(P_5P_6i)$ and $(P_6P_4i)$ in the same way and dropping the common factor (123) we have from (6)

$$\alpha_1 = \beta_4\frac{(567)\,(235)\,(236)\,(516)}{(P_5P_6P_7)} + \beta_5\frac{(647)\,(236)\,(234)\,(614)}{(P_6P_4P_7)} + \beta_6\frac{(457)\,(234)\,(235)\,(415)}{(P_4P_5P_7)} \tag{8}$$

From (4) we see that $A_{11} = (\beta_2\beta_3\alpha_1)$ if we neglect the factor (123). We therefore have from (8)

$$A_{11} = (234)\,(235)\,(236)\left\{\frac{(567)\,(516)}{(P_5P_6P_7)} + \frac{(647)\,(614)}{(P_6P_4P_7)} + \frac{(457)\,(415)}{(P_4P_5P_7)}\right\} \tag{9}$$

Similarly $A_{21} = (\beta_3\beta_1\alpha_1)$ giving

$$A_{21} = \sum_{456}\frac{(314)\,(567)\,(235)\,(236)\,(516)}{(P_5P_6P_7)} \tag{10}$$

where the terms of the sum are obtained from one another by cyclic advancement of the numbers 4, 5, 6. Also $A_{31} = (\beta_1\beta_2\alpha_1)$, giving

$$A_{31} = \sum_{456}\frac{(124)\,(567)\,(235)\,(236)\,(516)}{(P_5P_6P_7)} \tag{11}$$

## 3. IDENTITIES ON WHICH DEPENDS THE SIMPLIFICATION OF THE A'S.

The other six $A$'s are at once obtained from the first three by cyclic advancement of the numbers 1, 2, 3. Before doing this however, we may simplify the expressions just obtained, by means of identical relations connecting the factors which enter the numerators with the determinants which occur in the denominators.

The denominator $(P_4P_5P_7)$ is the determinant of the coefficients of the three vectors $P_4$, $P_5$, and $P_7$, and aside from a factor $(123)^2$ is equal, by C. Q. V. page 384, to the expression $C_{1234}(5, 7)$, which vanishes when the six vectors lie on a quadric cone. For our present purpose we may use a simpler notation and write

$$(P_4P_5P_7) = (123)^2\,C_6 \tag{12}$$

by designating the omitted vector. The other denominators may be transformed in a similar manner.

Consider now the identity

$$F\beta_1C_1 + F\beta_2C_2 + \cdots + F\beta_7C_7 = 0 \qquad (13)$$

where $F\beta_1$ is any quadratic function of $\beta_1$, $F\beta_2$ is the same function of $\beta_2$, etc.[3] If we let $F\lambda = (\lambda\beta_1\mu)\,(\lambda\beta_1\nu)$ we shall have $F\beta_1 = 0$ and the identity becomes

$$(\beta_2\beta_1\mu)\,(\beta_2\beta_1\nu)C_2 + (\beta_3\beta_1\mu)\,(\beta_3\beta_1\nu)C_3 + \cdots + (\beta_7\beta_1\mu)\,(\beta_7\beta_1\nu)C_7 = 0 \qquad (14)$$

which by the notation already employed may be written

$$\cdots + (21\mu)\,(21\nu)C_2 + (31\mu)\,(31\nu)C_3 + \cdots\,(71\mu)\,(71\nu)C_7 = 0 \qquad (15_1)$$

and similarly by letting $F\lambda = (\lambda\beta_2\mu)\,(\lambda\beta_2\nu)$,

$$(12\mu)\,(12\nu)C_1 + \cdots + (32\mu)\,(32\nu)C_3 + \cdots(72\mu)\,(72\nu)C_7 = 0 \qquad (15_2)$$

Proceeding thus we obtain a set of seven equations of which the last is

$$(17\mu)\,(17\nu)C_1 + (27\mu)\,(27\nu)C_2 + \cdots + (67\mu)\,(67\nu)C_6 + \cdots \qquad (15_7)$$

If we multiply these equations respectively by $C_1, C_2 \cdots C_7$ and add, we note that each term of the sum is of the form $(12\mu)\,(12\nu)\,C_1C_2$ and that each such term occurs twice, since $(12\mu)\,(12\nu) = (21\mu)\,(21\nu)$. Cancelling the factor 2 we thus have the new identity

$$\Sigma(12\mu)\,(12\nu)C_1C_2 = 0 \qquad (16)$$

where the left side contains as many terms as pairs can be chosen from the numbers one to seven, that is 21 terms.

So far $\mu$ and $\nu$ are any vectors whatever. Now let $\mu = \beta_1$ and $\nu = \beta_7$ causing all terms containing $C_1$ or $C_7$ to vanish. The remaining ten terms may be arranged as follows,

$$\begin{aligned}
&(561\ (567)C_5C_6 + (641)\,(647)C_6C_4 + (451)\,(457)C_4C_5 \\
&+ C_2[(241)\,(247)C_4 + (251)\,(257)C_5 + (261)\,(267)C_6] \\
&+ C_3[(341)\,(347)C_4 + (351)\,(357)C_5 + (361)\,(367)C_6] \\
&\qquad + (231)\,(237)C_2C_3 = 0 \qquad (17)
\end{aligned}$$

Returning to (13) and putting $F\lambda = (2\lambda 1)\,(2\lambda 7)$ we have, since the first, second, and seventh terms vanish,

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[3] This identity may be proved by noting that the left side is quadratic in $\beta_7$ and vanishes when $\beta_7$ coincides with any one of the other six vectors, hence vanishes identically. For a more detailed consideration of the C's see "An identity connecting seven vectors," Proc. Royal Soc. Edinburgh, **40**, Part II (No. 14), June 1920.

$$(231)\,(237)C_3+(241)\,(247)C_4+(257)\,(251)C_5+(261)\,(267)C_6=0 \quad (18)$$

That is, the expression in brackets in the second line of (17) is equal to $-(231)\,(237)C_3$. In a similar manner, the expression in brackets in the third line of (17) is equal to $-(231)\,(237)C_2$. Therefore (17) becomes

$$(561)\,(567)C_5C_6+(647)\,(641)C_6C_4+(451)\,(457)C_4C_5 = (231)\,(237)C_2C_3 \quad (19)$$

## 4. SIMPLIFICATION OF THE A'S.

Return now to our expression (9) for $A_{11}$, transform the denominators as in (12), and simplify,

$$A_{11} = (234)\,(235)\,(236)\,\frac{[-(561)\,(567)C_5C_6-(641)\,(647)C_6C_4-(451)\,(457)C_4C_5]}{(123)^2C_4C_5C_6}$$

$$= -(234)\,(235)\,(236)\,(237)\,\frac{C_2C_3}{(123)C_4C_5C_6}, \text{ by (19).} \quad (20)$$

Since it is evident that all the $A$'s will have a common denominator this may be rejected, and we may write as the value of $A_{11}$

$$A_{11} = (234)\,(235)\,(236)\,(237)C_2C_3 \quad (21)$$

It thus appears that $A_{11}$ vanishes when either of the four axes $\beta_4$, $\beta_5$, $\beta_6$, or $\beta_7$ lies in the plane of $\beta_2$, $\beta_3$. The quadratic vector is assumed irreducible, hence $C_2$ and $C_3$ do not vanish, as was proved in C. Q. V.

Considering next the simplification of $A_{21}$, we may first add the three terms, and reject the same factor as above. This gives

$$A_{21} = -(123)^{-1}\sum_{456}(314)\,(567)\,(561)\,(235)\,(236)C_5C_6 \quad (22)$$

From the fundamental identity (13), letting $F\lambda = (\lambda 67)\,(23\lambda)$,

$$(567)\,(235)C_5+(467)\,(234)C_4+(167)\,(231)C_1=0 \quad (23)$$

and by letting $F\lambda = (4\lambda 7)\,(23\lambda)$,

$$(457)\,(235)C_5+(467)\,(236)C_6+(417)\,(231)C_1=0 \quad (24)$$

It is now easy to eliminate $C_5$ from the last three identities. The terms in $C_6C_4$ cancel at the same time, giving

$$A_{21} = + (314)(561)(236)(167)C_1C_6 + (316)(451)(234)(417)C_1C_4 \quad (25)$$

For $A_{31}$ in a precisely similar manner,

$$A_{31} = + (124)(561)(236)(167)C_1C_6 + (126)(451)(234)(417)C_1C_4 \quad (26)$$

It is evident that expressions of similar form might have been found by eliminating either $C_4$ or $C_6$ instead of $C_5$.

The other six $A$'s may be found from (21), (25), and (26) by advancing the numbers 1, 2, 3, cyclically.

It is also evident that, by means of identities of the same form as (23) and (24), which connect any three C's with one another, we may express any one of the A's in terms of any pair of C's we wish, the coefficients being composed of scalar products of three axes.

## 5. Application to Cases of Coplanarity of Axes.

As a first illustration of the utility of the $A$'s, we may inquire what simplification takes place in the form of a quadratic vector when a set of three axes become coplanar,— a question intimately related to the problem of finding particular solutions of differential equations.[4]

One obvious answer is that if the three coplanar axes be numbered 4, 5, and 7, the general expression (5) loses its third term and becomes a binomial vector; but the form (5) implies a knowledge of all the axes, while the data of the problem frequently furnish only three axes,— for example, quadratic point transformations are often specified in terms of the singular points, which are our three axes $\beta_1$, $\beta_2$, $\beta_3$ such that $F\beta$ is null. Finding the remaining four axes and constructing the general expression (5) would then require, in general, the solution of an equation of the fourth degree. I propose to show that, if we have the quadratic vector in the form (1), or its equivalent (2), it is never necessary to find the other four axes in order to detect the existence of coplanarity among the axes; and, conversely, we may, by giving proper values to the $A$'s, immediately construct a quadratic vector possessing any required relations of coplanarity among the axes.

Taking the latter problem first, as being the simpler, if it be merely required *that one set of axes be coplanar,* we may let either of the three scalars $A_{11}$, $A_{22}$, or $A_{33}$ be zero. This is at once evident from the form of (21). For example, if $A_{11}$ is zero, one of the four axes $\beta_4$, $\beta_5$, $\beta_6$, $\beta_7$ must lie in the plane $\beta_2$, $\beta_3$. By a proper choice of the vector $\delta$ in (1), the quadratic vector can then be reduced to a binomial in the vectors $\beta_2$, $\beta_3$, the four diplanar axes becoming zeros, the value of $\delta$ being unique.

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[4] See note to C. Q. V. page 385.

If it be required to have *two sets of three coplanar axes* one way is to let $A_{11}$ and $A_{22}$ both vanish. But we have another equally easy way; if we let $A_{21}$ and $A_{32}$ both vanish, we shall have two sets of coplanar axes, $\beta_1$ being common to both sets. To prove this, note that, by (25) and (26), if $A_{21} = A_{31} = 0$, we have two linear equations in the quantities

$$(561)\,(236)\,(167) \quad \text{and} \quad (451)\,(234)\,(417) \tag{27}$$

the determinant of the coefficients being (314) (126) − (316) (124), which is the same as (123) (146). Now (123) is different from zero by hypothesis, for in assuming the form (2) we assume $\beta_1$, $\beta_2$, and $\beta_3$ to be diplanar;[5] on the other hand (146) may well be zero; if so we may write $\beta_1 = m\beta_4 + n\beta_6$ and either of the two equations reduces at once to

$$m\,(236)C_6 - n(234)C_4 = 0$$

by cancelling factors which cannot vanish if the quadratic vector is irreducible. This is the same as

$$(176)\,(236)C_6 + (174)\,(234)C_4 = 0$$

which by an identity of the form (23) gives

$$(175)\,(235) = 0.$$

Now (235) cannot vanish along with (146) hence (175) = 0, that is, we have two sets of coplanar axes having $\beta_1$, in common.

If the determinant in question does not vanish, the quantities (27) must both vanish. Remembering that we cannot have two distinct sets of coplanar axes nor four coplanar axes, if the quadratic vector is irreducible, we see by inspection that neither (234) nor (236) can vanish. Hence we must have the same case as above, viz. two sets including $\beta_1$.

If it be required to have *three sets of coplanar axes* it is now easy to pick out two quite different cases:

1°. We may let $A_{11}$, $A_{22}$, and $A_{33}$ all vanish. Three of the four axes $\beta_4$, $\beta_5$, $\beta_6$, $\beta_7$, lie, respectively, in the faces of the triedron whose edges are $\beta_1$, $\beta_2$, $\beta_3$.

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[5] It was shown in C. Q. V. that a quadratic vector possesses two distinct sets of three diplanar axes except in certain special cases.

2°. We may let $A_{22}$ and $A_{33}$ vanish and let $A_{23} = A_{32}$. The three sets of coplanar axes have $\beta_1$ in common.

The differential equations corresponding to these two cases are of very unlike character. In 1° the existence of the coplanar sets is evident from the form of the $A$'s. To prove it for 2° we may let the vector $\delta$ of (1) be expanded thus,

$$\delta S\beta_1\beta_2\beta_3 = a_1 V\beta_2\beta_3 + a_2 V\beta_3\beta_1 + a_3 V\beta_1\beta_2 \tag{28}$$

and since we have identically

$$\rho = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 \tag{29}$$

we shall have

$$S\delta\rho = a_1 x_1 + a_2 x_2 + a_3 x_3 \tag{30}$$

By expanding $V\varphi\rho\theta\rho$ as in (2), using the values of the $\alpha$'s from (4), and $S\delta\rho$ from (30), the fundamental equation (1) may be expanded in the frequently useful form

$$\begin{aligned} F\rho = \ & \beta_1[\qquad A_{11}x_2x_3 + (a_3 + A_{12})x_3x_1 + (a_2 + A_{13})x_1x_2 + a_1x_1^2] \\ & + \beta_2[(a_3 + A_{21})x_2x_3 + \qquad A_{22}x_3x_1 + (a_1 + A_{23})x_1x_2 + a_2x_2^2] \\ & + \beta_3[(a_2 + A_{31})x_2x_3 + (a_1 + A_{32})x_3x_1 + \qquad A_{33}x_1x_2 + a_3x_3^2] \end{aligned} \tag{31}$$

This is a way of expressing a quadratic vector which is always possible except in the very special cases examined in C. Q. V. where a set of three diplanar axes cannot be found; this expression depends on the twelve constants which occur explicitly, and on the three directions $\beta_1$, $\beta_2$, $\beta_3$ of the diplanar axes, equivalent to six more scalars. If now in addition to the three conditions $A_{22} = A_{33} = 0$, $A_{23} = A_{32}$ just assumed, we take

$$a_1 = -A_{32}, \qquad a_2 = 0, \qquad a_3 = 0$$

the quadratic vector takes the form

$$F\rho = \beta_1[A_{11}x_2x_3 + A_{12}x_3x_1 + A_{13}x_1x_2 - A_{23}x_1^2] + (\beta_2 A_{21} + \beta_3 A_{31})x_2x_3 \tag{32}$$

But this is a binomial; it must, therefore, by the reasoning of C. Q. V. page 384, possess three axes in the plane of the vector coefficients $\beta_1$ and $(\beta_2 A_{21} + \beta_3 A_{31})$. But we already know it possesses three axes in the planes $\beta_1$, $\beta_2$ and $\beta_1$, $\beta_3$, respectively. It therefore has three coplanar sets with $\beta_1$ in common. For brevity an axis common to three coplanar sets may be called a *central axis.*

## 6. PROPERTIES OF A QUADRATIC VECTOR POSSESSING A CENTRAL AXIS.

The differential equations determined by this type of vector have been studied by Darboux[6]; the connection between his viewpoint and that of vectorial algebra is established by certain theorems now to be proved.

**Theorem 1.** If a quadratic vector has a central axis, it may be thrown into binominal form in three distinct ways, a proper choice of the vector $\delta$ being made in each case.

One way has already been shown in (32), namely for the value of $\delta$ given by $a_1 = -A_{32}$, $a_2 = a_3 = 0$.

A second way, by inspection of (31), is to take $a_3 = 0$, $a_1 = -A_{32}$, and $a_2 = -A_{31}$. Since $A_{33}$ is zero by hypothesis, the component along $\beta_3$ vanishes.

Similarly, if $\delta$ is determined by making $a_2 = 0$, $a_1 = -A_{23}$, $a_3 = -A_{21}$, the component along $\beta_2$ vanishes. The theorem is thus proved.

**Theorem 2.** If a quadratic vector has a central axis, the three values of $\delta$ by virtue of which we may pass from one binomial form to another are all perpendicular to the central axis.

**Proof.** These values of $\delta$ are the respective differences of the values occurring in the proof of theorem 1. They must therefore be coplanar. To determine their values explicitly, let the binomial forms be $F_1$, $F_2$, and $F_3$ in the order above given. Let the values of $\delta$ which changes $F_1$ into $F_2$, $F_2$ into $F_3$, and $F_3$ into $F_1$ be, in order, $\delta_1$, $\delta_2$, and $\delta_3$. Subtracting values of the $a$'s as above found we have, for $\delta_1$, $a_1 = 0$, $a_2 = -A_{31}$, $a_3 = 0$; for $\delta_2$, $a_1 = 0$, $a_2 = +A_{31}$, $a_3 = -A_{21}$; for $\delta_3$, $a_1 = 0$, $a_2 = 0$, $a_3 = +A_{21}$. Since $a_1$ is zero in all three cases, the theorem is proved.

**Theorem 3.** If a quadratic vector having a central axis be thrown into either of the three binomial forms, a set of *rectangular* components $X$, $Y$, $Z$, can be found in terms of rectangular coördinates $x$, $y$, $z$, such that both $X$ and $Y$ are independent of $z$.

**Proof.** With the notation used in the proof of theorem 2, let the quadratic vector be in the binomial form $F_2$. Let $\beta_1 = k$, and let $i$ and $j$ be any two unit vectors such that $i$, $j$, $k$ forms a rectangular unit system. Since $\delta_2$ is perpendicular to $i$ we may write $\delta_2 = ci + c'j$. Also $\rho = ix + jy + kz$. Hence $\rho S\delta_2\rho = -i(cx^2 + c'xy) - j(cxy + c'y^2) +$ terms in $k$. It was shown that $F_3 = F_2 + \rho S\delta_2\rho$. Now $F_2$ contained no component along $\beta_3$. By adding $\rho S\delta_2\rho$ we removed the component along $\beta_2$. But $\rho S\delta_2\rho$ does not contain $z$. Therefore

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[6] See note to C. Q. V. page 375.

$F_3$ contains a component along $\beta_1$ or $k$, and terms free from $z$. Similarly it may be shown that $F_1$ and $F_2$ may be expressed according to theorem 3.

**Theorem 4.** Conversely, if a quadratic vector be expressed in rectangular coördinates, and if its components $X$ and $Y$ are free from $z$, it has a central axis.

**Proof.** The two-dimensional quadratic vector $iX + jY$ has three axes. Call them $\gamma_1$, $\gamma_2$. and $\gamma_3$. Take $i$ along $\gamma_1$ (by a rotation of axes in the $xy$ plane if necessary, which cannot alter the fact that $X$ and $Y$ are free from $Z$). We may then write

$$iX + jY = i(ax^2 + bxy) + j(a'x^2 + b'xy + c''y^2)$$

It is evident from the form of the term $\rho S\delta\rho$ that we may now, by a proper choice of $\delta$, remove the component along $i$, namely by taking $\delta = ai + bj$. The given quadratic vector will then have the form $jY + kZ$, hence possesses three axes in the plane perpendicular to $\gamma_1$. Similarly we may show that the given quadratic vector has three axes in the plane perpendicular to $\gamma_2$ and similarly for $\gamma_3$. But $k$ is the common line of intersection of these three planes, and is itself an axis, since $X$ and $Y$ vanish when $x$ and $y$ are both zero. That is, $k$ is a central axis.

**Theorem 5.** The vectors $\delta_1$, $\delta_2$, $\delta_3$, which convert, respectively, $F_1$ into $F_2$, $F_2$ into $F_3$ and $F_3$ into $F_1$, are axes of the two-dimensional vector $iX + jY$.

**Proof.** Inspection of the three $\delta$'s obtained in proving theorem 2 shows that they are perpendicular, respectively, to $F_3$, $F_1$, and $F_2$. But in the proof of theorem 4 it appeared that the three $\gamma$'s have the same property. Hence the directions of the $\delta$'s coincide with the respective $\gamma$'s.

**Theorem 6.** If a quadratic vector $V\varphi\rho\theta\rho + \rho S\delta\rho$ has a central axis, and if a proper choice of rectangular coördinates be made, a value can be found for $\delta$ (which does not alter the axes), such that the differential equation $\dfrac{dy}{dx} = \dfrac{Y}{X}$ takes the Riccati form.

**Proof.** As in the proof of theorem 4 we may obtain the quadratic vector in the form

$$i(ax^2 + bxy) + j(a'x^2 + b'xy + c'y^2) + kZ$$

By taking $\delta = bj$ and adding the term $\rho S\delta\rho$ this becomes

$$iax^2 + j[a'x^2 + b'xy + (c' - b)y^2] + k[Z - byz]$$

which proves the theorem.

The connection between the theory of quadratic vectors and Darboux's treatment of differential equations is now fully established. To continue the study of the above differential equation would be merely to repeat Darboux's work.

To indicate a quite different application of the present theory we may note the following,—

**Theorem 7.** If a quadratic 1 : 1 point transformation be defined by the equations $x_1^1 = X_1(x_1, x_2, x_3)$, $x_2^1 = X_2(x_1, x_2, x_3)$, $x_3^1 = X_3(x_1, x_2, x_3)$, and if the four fixed points of the transformation be situated as follows: the singular points being A, B, C, two fixed points lie on a straight line through $A$, the other two lie respectively on $AB$ and $AC$; — the transformation can be written in the form

$$x' = tx, \quad y' = Y(x, y,) + ty, \quad z' = Z(x, y, z)$$

*where t is a linear function of x, y, z.*

This theorem is, of course, an immediate consequence of theorems 2 and 3, stated in the language of point transformations.

## 7. Vectors with Four Sets of Coplanar Axes.

Continuing the study of the simplifications which occur in the form of a quadratic vector when sets of coplanar axes exist, let it be required to have four such sets. Here, again, we shall evidently have two cases, according as we have a central axis or not.

If there is to be no central axis, we may begin with case 1° of Art. 5, letting the six quantities $A_{11}$, $A_{22}$, $A_{33}$, and $a_1$, $a_2$, $a_3$, all vanish. The quadratic vector then takes the form

$$F\rho = \beta_1(A_{12}x_3x_1 + A_{13}x_1x_2) + \beta_2(A_{21}x_2x_3 + A_{23}x_1x_2) + \beta_3(A_{31}x_2x_3 + A_{32}x_3x_1); \qquad (33)$$

it is evident that three of the axes may be taken as follows,—

$$\beta_4 = \beta_2A_{21} + \beta_3A_{31}; \; \beta_5 = \beta_3A_{32} + \beta_1A_{12}; \; \beta_6 = \beta_1A_{13} + \beta_2A_{23}. \qquad (34)$$

To find $\beta_7$ we note that, by C. Q. V. page 385, this axis is perpendicular to the three values of $\delta$ by which we may pass from one binomial form of $F\rho$ to another. Thus by an easy calculation

$$\beta_7 = \sum_{123} [\beta_1(A_{12}A_{31} + A_{21}A_{13} - A_{21}A_{31})] \qquad (35)$$

We now have three coplanar sets given by $(234) = (315) = (126) = 0$. We may not include $\beta_7$ in a fourth set if the quadratic vector is to be irreducible and have no central axis. We must therefore take $(456) = 0$ which evidently implies the vanishing of the determinant

$$\begin{vmatrix} 0, & A_{21}, & A_{31} \\ A_{12}, & 0, & A_{32} \\ A_{13}, & A_{23}, & 0 \end{vmatrix} \tag{36}$$

*Six* axes of the quadratic vector will now be along the lines of intersection of four planes. This fact suggests an expression for the vector $F\rho$ which shall be symmetrical in these six axes: for consider the vector $V\gamma_1\gamma_2S\gamma_3\rho S\gamma_4\rho$, where the $\gamma$'s are taken at right angles to the four planes respectively; all six lines of intersection of the four planes with each other are axes of this vector term, and the same will be true no matter in what order we write the four subscripts. If, then, $a_{12}$, $a_{13}$, $a_{23}$, $a_{14}$, $a_{24}$, and $a_{34}$ be any six scalars, the vector sum

$$F\rho = a_{12}V\gamma_1\gamma_2S\gamma_3\rho S\gamma_4\rho + a_{13}V\gamma_1\gamma_3S\gamma_2\rho S\gamma_4\rho + \cdots + a_{34}V\gamma_3\gamma_4S\gamma_1\rho S\gamma_2\rho \tag{37}$$

will be a quadratic vector of the type under consideration, namely it will have six axes in the directions $V\gamma_1\gamma_2$, $V\gamma_1\gamma_3, \cdots V\gamma_3\gamma_4$. This form of $F\rho$ has a number of interesting properties easily proved by methods already exemplified. As instances,—

1. If $\rho S\delta\rho$ be added, there are four values of $\delta$ which render $F\rho$ a binomial. One value is

$$\delta_1 = -[a_{23}(123)\gamma_4 + a_{24}(124)\gamma_3 + a_{34}(134)\gamma_2] \tag{38}$$

and the others are of similar form.

2. If $\delta_1$, $\delta_2$, $\delta_3$, $\delta_4$ be the values of $\delta$ just mentioned, the differences $\delta_1 - \delta_2$, $\delta_1 - \delta_3$, $\delta_2 - \delta_3$ etc. are all perpendicular to the seventh axis. Hence the direction of this axis is easily calculated.

3. The quadratic vector (37) possess some properties closely analogous to those of the general linear vector function; for the latter may be written

$$g_{23}V\lambda_2\lambda_3S\lambda_1\rho + g_{31}V\lambda_3\lambda_1S\lambda_2\rho + g_{12}V\lambda_1\lambda_2S\lambda_3\rho \tag{39}$$

and has for axes $V\lambda_2\lambda_3$, etc.

4. If $F\rho$ has the form (37) the cubic vector $V\rho F\rho$ takes the form $\Sigma b_1\gamma_1S\gamma_2\rho S\gamma_3\rho S\gamma_4\rho$ where $b_1 = -a_{12} - a_{13} - a_{14}$ and $b_2$, $b_3$, $b_4$ are easily found. The sum of the $b$'s is zero.

5. If $V\rho F\rho$ be divided by the product $S\gamma_1\rho S\gamma_2\rho S\gamma_3\rho S\gamma_4\rho$ the quotient is an irrotational vector; it is normal to the family of cones $(S\gamma_1\rho)^{b_1}(S\gamma_2\rho)^{b_2}(S\gamma_3\rho)^{b_3}(S\gamma_4\rho)^{b_4} = \text{const.}$

If, on the other hand, *$F\rho$ is to have a central axis,* and four coplanar sets, we may begin with case 2° of Art. 5 and let $A_{11}$ be zero. The only new property introduced, so far as I am aware, is that the system $\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$ can now be fully solved by quadratures. The proof may be carried out by the aid of the theorems already established.

## 8. OTHER SPECIAL TYPES.

There are but two remaining types depending on coplanarity of axes. These are first, a vector sharing the properties of the two vectors of the last article, possessing, therefore, five coplanar sets, and having two central axes; second, a vector having six coplanar sets and four central axes. They are easily obtained from the types which precede. The vector having four central axes bears an interesting resemblance to the *self-conjugate* linear vector function. The investigation may be left to the reader.

## 9. THE CONVERSE PROBLEM.

In what precedes it has been shown vectors of assigned type may be written down by giving proper values to the $A$'s. I now propose to examine the converse problem: *given any values of the nine $A$'s, to test whether the vector possesses three coplanar axes.*

Evidently, if the vector should happen to fall under one of the forms which have been explicitly written out, we would at once recognize its type; but these vectors have been merely "normal forms" of their respective types, hence not sufficient to serve as tests for comparison.

The necessary and sufficient condition for coplanarity of axes of the vector (1) may be obtained as follows. A set of three coplanar axes exists if, and only if, a value of $\delta$ can be found which makes the right side of (1) a binomial, (proved C. Q. V. p. 384). Let $\gamma$ be normal to the plane of this binomial. Then

$$S\gamma F\rho = 0 \tag{40}$$

identically. Let $\gamma$ be expanded as

$$\gamma = p_1V\beta_2\beta_3 + p_2V\beta_3\beta_1 + p_3V\beta_1\beta_2 \tag{41}$$

Let this value of $\gamma$, and the expanded form of $F\rho$ from (31) be substituted in (40). By equating to zero the coefficients of $x_1^2$, $x_1x_2$, etc. we obtain six equations,

$$\begin{aligned} p_1a_1 &= 0, \quad & p_1A_{11} + p_2(a_3 + A_{21}) + p_3(a_2 + A_{31}) &= 0 \\ p_2a_2 &= 0, \quad & p_1(a_3 + A_{12}) + p_2A_{22} + p_3(a_1 + A_{32}) &= 0 \\ p_3a_3 &= 0, \quad & p_1(a_2 + A_{13}) + p_2(a_1 + A_{23}) + p_3A_{33} &= 0 \end{aligned} \tag{42}$$

In order that $\gamma$ may exist, not all the $p$'s can vanish. We shall then have three cases, according as three, two, or one, of the $a$'s shall be zero.

Case 1°. $a_1 = a_2 = a_3 = 0$. Values can be found for the $p$'s if, and only if, the determinant of the $A$'s vanishes. The vectors $\alpha$ of the right side of (2) are then coplanar, and three of the four axes $\beta_4$, $\beta_5$, $\beta_6$, $\beta_7$ lie in this plane.

Case 2°. Two only of the $a$'s are zero. Suppose $a_2 = a_3 = 0$, with $a_1$ not zero. Then $p_1 = 0$. The six linear equations reduce to

$$p_2A_{21} + p_3A_{31} = 0 \tag{43}$$

$$p_2A_{22} + p_3(a_1 + A_{32}) = 0 \tag{44}$$

$$p_2(a_1 + A_{23}) + p_3A_{33} = 0 \tag{45}$$

Now $p_2$ and $p_3$ cannot both vanish if $\gamma$ exists. Hence

$$a_1A_{21} + A_{21}A_{32} - A_{31}A_{22} = 0 \tag{46}$$

$$a_1A_{31} + A_{31}A_{23} - A_{21}A_{33} = 0 \tag{47}$$

$$a_1^2 + a_1(A_{23} + A_{32}) + A_{32}A_{23} - A_{22}A_{33} = 0 \tag{48}$$

From (46) and (47) we have

$$\begin{matrix} 0, & A_{21}, & A_{31} \\ -A_{21}, & A_{22}, & A_{32} \\ A_{31}, & A_{23}, & A_{33} \end{matrix} = 0 \tag{49}$$

If $A_{21}$ and $A_{31}$ are not both zero, $a_1$ can be found from (46) or from (47). Also if $A_{21}$ and $A_{31}$ are not both zero (48) is a consequence of (46) and (47). Since $p_1$ is zero, the plane of the coplanar axes contains $\beta_1$.

If $A_{21}$ and $A_{31}$ are both zero, two values of $a_1$ are found from (48), in general distinct, and there are two coplanar sets with $\beta_1$ in common.

In any case, if the determinant (49), or either of the similar determinants obtained from it by advancing subscripts, is zero, there is at least one set of coplanar axes.

Case 3°. Only one of the $a$'s is zero. Suppose $a_3 = 0$, with $a_1$ and $a_2$ not zero. Then $p_1 = p_2 = 0$. The six equations reduce to

$$\begin{aligned} a_2 + A_{31} &= 0 \\ a_1 + A_{32} &= 0 \\ A_{33} &= 0 \end{aligned}$$

Hence if $A_{33} = 0$ the $a$'s can be determined and a coplanar set exists. Similarly for $A_{11}$ and $A_{22}$, in agreement with a former result.

*Summary of tests for coplanarity of axes.*

*The quadratic vector* (31) *possesses three axes in the same plane when, and only when, one of these seven conditions holds: the vectors* $\alpha_1$, $\alpha_2$, $\alpha_3$ *defined by* (*4*) *are coplanar; one of the three determinants of the form* (*49*) *vanishes; or one of the constants* $A_{11}$, $A_{22}$, $A_{33}$ *vanishes.*

## 10. APPLICATION TO IRROTATIONAL VECTORS.

As a second illustration of the utility of the constants $A_{11}$ etc., I propose to examine the properties of a quadratic vector under the requirement that it be irrotational, i.e. its "curl" shall be zero or

$$V\nabla F\rho = 0 \tag{50}$$

The significance of this equation on the physical side is well known. To see the algebraic aspect of the problem we may recall that, as was shown by Hamilton,[7] any *linear* vector function $\varphi\rho$ may be written as the sum of two terms thus,

$$\varphi\rho = \omega\rho + V\epsilon\rho \tag{51}$$

where $\epsilon$ is a vector; $\omega$ is self-conjugate, irrotational, and has its three axes at right angles to each other. It is natural to enquire what restriction is imposed on the axes in the case of an irrotational quadratic vector.

The scope of the enquiry will appear from the following:

**Theorem 8.** If a quadratic vector can be made irrotational without altering its axes, its curl is of the form $V\delta\rho$.

The proof is evident from the identity

$$V\nabla(F\rho + \rho S\delta\rho) = V\nabla F\rho + V\rho\delta \tag{52}$$

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[7] *Elements*, Art. 349; 2nd Ed. p. 492.

for if $F\rho$ can be made irrotational by adding $\rho S\delta\rho$, (the only form of term which leaves the axes unaltered), the right member must vanish, and we have $V\nabla F\rho = V\delta\rho$; hence the theorem. The converse is equally obvious,—

**Theorem 9.** If a quadratic vector has its curl of the form $V\delta\rho$ it can be made irrotational by adding the term $\rho S\delta\rho$.

## 11. CONDITIONS THAT THE CURL SHALL BE OF THE FORM $V\delta\rho$.

I shall now show that, in general, a set of five scalar equations exists which are necessary and sufficient that the curl of a quadratic vector be of the form $V\delta\rho$. These will appear as equations connecting the nine $A$'s, and involving also the axes $\beta_1$, $\beta_2$, $\beta_3$. Since the $A$'s have been obtained as functions of the axes, these equations impose restrictions on the axes of an irrotational quadratic vector.

Taking (31) we let $a_1 = a_2 = a_3 = 0$, which is equivalent to neglecting the term $\rho S\delta\rho$. We then operate by $V\nabla$. Now $\nabla x_1 \cdot S\beta_1\beta_2\beta_3 = -V\beta_2\beta_3$ and similarly for $x_2$ and $x_3$. Hence easily (using the $\alpha$'s from (4)),

$$V\nabla F\rho \cdot S\beta_1\beta_2\beta_3 = \sum_{123} V\alpha_1(x_2 V\beta_1\beta_2 + x_3 V\beta_3\beta_1) \qquad (53)$$

The right side of this equation is a linear vector function of $\rho$, which we may call $\theta\rho$. Putting for the $x$'s their values, and arranging the order of terms we may write

$$\theta\rho = \sum_{123} (V\alpha_2 V\beta_1\beta_2 + V\alpha_3 V\beta_3\beta_1) S\rho\beta_2\beta_3 \qquad (54)$$

which must be of the form $V\delta\rho$. This is the same as saying that the self-conjugate part of $\theta$ must vanish, or that $\theta + \theta' = 0$, or again that $S\rho\theta\rho$ must vanish for all values of $\rho$. In general the vanishing of a self-conjugate linear vector function is equivalent to six scalar equations; but in this case we note that $S\nabla\theta\rho = \sum_{123} S(V\alpha_2 V\beta_1\beta_2 + V\alpha_3 V\beta_3\beta_1) V\beta_2\beta_3 = 0$ identically, hence the six equations are not independent.

A simple way to set up the six equations in explicit form is $S\beta_1\theta\beta_1 = 0$, $S\beta_2\theta\beta_2 = 0$, $S\beta_3\theta\beta_3 = 0$, $S\beta_2\theta\beta_3 + S\beta_3\theta\beta_2 = 0$, $S\beta_3\theta\beta_1 + S\beta_1\theta\beta_3 = 0$, and $S\beta_1\theta\beta_2 + S\beta_2\theta\beta_1 = 0$. Since by hypothesis $\beta_1$, $\beta_2$, and $\beta_3$ are not coplanar these are sufficient to make $\theta + \theta' = 0$. By (54) the first and fourth of these equations are

$$S \cdot V\beta_1\alpha_2 V\beta_1\beta_2 + S\beta_1\alpha_3 V\beta_3\beta_1 = 0 \qquad (55)$$

$$S\cdot V\beta_2\alpha_1 V\beta_3\beta_1 + S\cdot V\beta_2\alpha_2 V\beta_2\beta_3 + S\cdot V\beta_3\alpha_3 V\beta_2\beta_3 + S\cdot V\beta_3\alpha_1 V\beta_1\beta_2 = 0 \qquad (56)$$

and the others are obtained by advancing subscripts.

Inspection of the scalar products which occur in these equations shows that irrotationality of the quadratic vector $F\rho$ is dependent on the relation of the $\alpha$'s to the two systems of vectors $\beta_1$, $\beta_2$, $\beta_3$ and $V\beta_2\beta_3$, $V\beta_3\beta_1$, $V\beta_1\beta_2$. The most natural procedure is to expand the latter system thus

$$\left.\begin{aligned} V\beta_2\beta_3\cdot S\beta_1\beta_2\beta_3 &= b_{11}\beta_1 + b_{21}\beta_2 + b_{31}\beta_3 \\ V\beta_3\beta_1\cdot S\beta_1\beta_2\beta_3 &= b_{12}\beta_1 + b_{22}\beta_2 + b_{32}\beta_3 \\ V\beta_1\beta_2\cdot S\beta_1\beta_2\beta_3 &= b_{13}\beta_1 + b_{23}\beta_2 + b_{33}\beta_3 \end{aligned}\right\} \qquad (57)$$

whence we have

$$\begin{aligned} b_{11} = S\cdot V\beta_2\beta_3 V\beta_2\beta_3 = S^2\beta_2\beta_3 - \beta_2^2\beta_3^2; \quad b_{21} = b_{12} &= S\cdot V\beta_2\beta_3 V\beta_3\beta_1 \\ &= \beta_3^2 S\beta_1\beta_2 - S\beta_2\beta_3 S\beta_3\beta_1 \qquad (57) \end{aligned}$$

and similarly for the other $b$'s. We may now introduce the expansions of the $\alpha$'s from (4) and of $V\beta_2\beta_3$ etc. from (57) and the six equations of form (55) and (56) become

$$\begin{aligned} b_{33}A_{22} - b_{22}A_{33} + b_{23}(A_{23} - A_{32}) &= 0 \qquad (58_1) \\ b_{11}A_{33} - b_{33}A_{11} + b_{31}(A_{31} - A_{13}) &= 0 \qquad (58_2) \\ b_{22}A_{11} - b_{11}A_{22} + b_{12}(A_{12} - A_{21}) &= 0 \qquad (58_3) \\ -b_{11}(A_{23} - A_{32}) + b_{12}(A_{31} + A_{13}) - b_{13}(A_{12} + A_{21}) &= 0 \qquad (58_4) \\ -b_{22}(A_{31} - A_{13}) + b_{23}(A_{12} + A_{21}) - b_{12}(A_{23} + A_{32}) &= 0 \qquad (58_5) \\ -b_{33}(A_{12} - A_{21}) + b_{13}(A_{23} + A_{32}) - b_{23}(A_{31} + A_{13}) &= 0 \qquad (58_6) \end{aligned}$$

Here we note that if the six equations be multipled, in order, by $b_{11}$, $b_{22}$, $b_{33}$, $b_{23}$, $b_{31}$, $b_{12}$, and the results added, the sum of the left members is identically zero; the six equations are not independent.

Since the $A$'s have already been determined as functions of the axes, the six equations (58) are necessary conditions which the axes must satisfy when the quadratic vector is irrotational.

We note further that, assuming $\beta_1$, $\beta_2$, and $\beta_3$ to be real, the scalars $b_{11}$, $b_{22}$, and $b_{33}$ are different from zero. These three scalars could all vanish only if $V\beta_2\beta_3$, $V\beta_5\beta_1$, and $V\beta_1\beta_2$ were all minimal vectors, i. e. imaginaries of null tensor; thus in general we may assume $b_{11}$ different from zero; and except in this very special case, therefore, $(58_1)$ is a consequence of the other five equations which, with the exception noted, are sufficient that the curl shall be of the form $V\delta\rho$.

## 12. INTERPRETATION OF THE EQUATIONS (58).

It is manifest that the equations (58) are bilinear: and that the set of six $b$'s are determined by the choice of three axes $\beta_1$, $\beta_2$ $b_3$, while the set of nine $A$'s govern the other four axes. I shall first show how various simple solutions of these equations may be written down, and shall then discuss the general solution as a linear function of these simple solutions.

*One solution* is seen by inspection to be: $A_{11}$, $A_{22}$, $A_{33}$, proportional to $b_{11}$, $b_{22}$, and $b_{33}$, with the other six $A$'s all zero. The resulting value of $F\rho$, which we may call $F_1\rho$, is, by (31),

$$F_1\rho = \beta_1 b_{11} x_2 x_3 + \beta_2 b_{22} x_3 x_1 + \beta_3 b_{33} x_1 x_2 \qquad (59_1)$$

which, of course, may be multiplied by a scalar constant. It is easy to check the result by operating with $\nabla$, using the values of the $b$'s from $(57_1)$, and showing that the expression is of the form $V\rho\delta$.

*A second solution* is equally obvious: let $A_{23} + A_{32}$, $A_{31} + A_{13}$, and $A_{12} + A_{21}$ be proportional to $b_{23}$, $b_{31}$ and $b_{12}$ with $A_{11} = A_{22} = A_{33} = 0$ and $A_{23} = A_{32}$, $A_{31} = A_{13}$, $A_{12} = A_{21}$, giving the solution

$$F_2\rho = \beta_1(b_{12}x_3x_1 + b_{13}x_1x_2) + \beta_2(b_{21}x_2x_3 + b_{23}x_1x_2) + \beta_3(b_{31}x_2x_3 + b_{32}x_3x_1) \qquad (59_2)$$

We note that this solution is null if $b_{23}$, $b_{31}$, and $b_{12}$ are all zero, that is, if the vectors $\beta_1$, $\beta_2$, and $\beta_3$ form a rectangular system. Let us suppose for the moment that one of the $b$'s, as $b_{23}$, is not zero.

These two solutions are linear in the $b$'s, and are not altered by advancing subscripts. A third solution, is found by assigning arbitrarily $A_{11} = A_{22} = A_{23} + A_{32} = 0$ and solving for the $A$'s in terms of the $b$'s. The result is

$$A_{33} = -2b_{31}b_{23},\ A_{31} = b_{23}b_{11},\ A_{13} = -b_{23}b_{11},\ A_{12} = b_{22}b_{11},$$
$$A_{21} = b_{22}b_{11},\ A_{23} = -b_{22}b_{31},\ A_{32} = +b_{22}b_{31},$$

which may be checked by substituting in (58). Whence by (31)

$$F_3\rho = \beta_1(b_{22}b_{11}x_3x_1 - b_{23}b_{11}x_1x_2) + \beta_2(b_{22}b_{11}x_2x_3 - b_{22}b_{31}x_1x_2)$$
$$+ \beta_3(b_{23}b_{11}x_2x_3 + b_{22}b_{31}x_3x_1 - 2b_{31}b_{23}x_1x_2) \qquad (59_3)$$

quadratic and unsymmetrical in the $b$'s.

A fourth solution may now be obtained by advancing subscripts, as also, of course, a fifth, which might be used in place of $F_2\rho$ in case all $b$'s with unequal subscripts vanish, but a simpler treatment of this case will be given below. (It is evident that we cannot in general have more than four linearly independent solutions for the $A$'s when the $b$'s are assigned.) Thus

$$\begin{aligned}F_4\rho = {} & \beta_1(-2b_{11}b_{31}x_2x_3 + b_{31}b_{22}x_3x_1 + b_{33}b_{12}x_1x_2) \\ & + \beta_2(-b_{31}b_{22}x_2x_3 + b_{33}b_{22}x_1x_2) + \beta_3(-b_{33}b_{12}x_2x_3 + b_{33}b_{22}x_3x_1)\end{aligned} \qquad (59_4)$$

It is furthermore clear, because the equations (58) are linear in the $A$'s, that any linear function of the solutions already obtained will be a solution, the $b$'s being taken as known constants, and, we will thus obtain the most general solution. For from the four solutions already written we may pick out the matrix of the four sets of values of $A_{11}$, $A_{22}$, $A_{33}$, namely

$$\begin{array}{ccc} b_{11}, & b_{22}, & b_{33} \\ 0, & 0, & 0 \\ 0, & 0, & -2b_{31}b_{23} \\ -2b_{12}b_{31}, & 0, & 0 \end{array}$$

which by inspection are linearly independent sets. Hence the four solutions $F_1$, $F_2$, $F_3$, $F_4$ are linearly independent.

Consider now the vector $F_1\rho$. Remembering that the axes are the vector solutions of the equation $V\rho F\rho = 0$, we expand $\rho$ as $\beta_1x_1 + \beta_2x_2 + \beta_3x_3$ and have these three equations to determine the axes of $F_1\rho$,

$$b_{33}x_1x_2^2 - b_{22}x_1x_3^3 = 0, \quad b_{11}x_2x_3^2 - b_{33}x_2x_1^2 = 0, \quad b_{22}x_3x_1^2 - b_{11}x_3x_2^2 = 0$$

Writing for brevity $c_1 = \sqrt{b_{11}}$, $c_2 = \sqrt{b_{22}}$, and $c_3 = \sqrt{b_{33}}$ we find easily the matrix of the coefficients of $\beta_1$, $\beta_2$, $\beta_3$ for the seven axes to be

$$\begin{array}{cccc} \beta_1: & 1 & 0 & 0 \\ \beta_2: & 0 & 1 & 0 \\ \beta_3: & 0 & 0 & 1 \\ \beta_4: & c_1 & c_2 & c_3 \\ \beta_5: & -c_1 & c_2 & c_3 \\ \beta_6: & c_1 & -c_2 & c_3 \\ \beta_7: & c_1 & c_2 & -c_3 \end{array}$$

that is, the fourth axis is given by $\beta_4 = c_1\beta_1 + c_2\beta_2 + c_3\beta_3$ etc. By inspection of this matrix we see the following relations of coplanarity among the axes: $(145) = (167) = (246) = (257) = (356) = (347) = 0$. The vector $F_1\rho$ therefore is of the type mentioned in Art. 8 having four central axes, namely $\beta_4$, $\beta_5$, $\beta_6$, $\beta_7$.

Finding the axes of $F_2\rho$ in a similar manner we have the matrix

$$\begin{array}{llll}
\beta_1: & 1 & 0 & 0 \\
\beta_2: & 0 & 1 & 0 \\
\beta_3: & 0 & 0 & 1 \\
\beta_4: & 0 & b_{12} & b_{13} \\
\beta_5: & b_{12} & 0 & b_{23} \\
\beta_6: & b_{13} & b_{23} & 0 \\
\beta_7: & b_{12}b_{13} & b_{12}b_{23} & b_{23}b_{31}
\end{array}$$

whence the relations of coplanarity $(126) = (234) = (315) = (147) = (257) = (367) = 0$, so that $F_2\rho$ belongs to the same type as $F_1\rho$, with central axes $\beta_1$, $\beta_2$, $\beta_3$, and $\beta_7$.

The vector $F_3\rho$ is of different type. The plane $x_3 = 0$ contains only the two axes $\beta_1$ and $\beta_2$. Besides $\beta_3$ we have the axes $(0, b_{22}, b_{23})$ and $(b_{11}, 0, b_{31})$ together with two imaginary axes in the plane

$$b_{22}b_{31}x_1 = b_{23}b_{11}x_2$$

where $x_2$ and $x_3$ satisfy the quadratic

$$b_{22}x_3^2 - b_{23}b_{22}x_2x_3 + 2b_{23}^2x_2^2 = 0.$$

We see that $F\rho$ is not necessarily of so restricted a type as $F_1\rho$.

## 13. CASE WHERE THREE GIVEN AXES FORM A RECTANGULAR SYSTEM.

Of special interest is the case where the three assigned axes $\beta_1$, $\beta_2$, $\beta_3$, are mutually perpendicular both because the four solutions (59) are no longer linearly independent, and because we might suspect here some greater analogy with linear vector functions. We now have $b_{12} = b_{23} = b_{31} = 0$ and the equations (58) have the evident solutions, (letting $i$, $j$, $k$ replace $\beta_1$, $\beta_2$, $\beta_3$, so that $b_{11} = b_{22} = b_{33} = 1$),

$$\begin{aligned}
F_1\rho &= iyz + jzx + kxy \\
F_2\rho &= jxy + kxz \\
F_3\rho &= kyz + ixy \\
F_4\rho &= ixz + jyz
\end{aligned}$$

which are linearly independent. The last three solutions are, to be sure, reducible, but linear combinations of them will not in general be so. Thus in all cases the general solution may be expressed as the sum of four simpler solutions.

## 14. APPLICATION TO CONSECUTIVE CHEMICAL PROCESSES.

As another illustration of the utility of the coefficients $A_{11}$ etc., it may be noted that an important class of chemical processes, namely, the type known as "consecutive," leads to a pair of differential equations of the form

$$\frac{dx_1}{dt} = P_1(x_1, x_2), \qquad \frac{dx_2}{dt} = P_2(x_1, x_2)$$

where $P_1$ and $P_2$ are quadratic polynomials, (not in general homogeneous), in the dependent variables $x_1$, $x_2$, but are not functions of $t$.

These equations cannot in general be solved by quadratures, but by a proper choice of the conditions of the process they may often be made integrable in this way, and the labor of solving by series avoided. We have merely to render the polynomials $P_1$ and $P_2$ homogeneous by introducing a third variable $x_3$ and write down the nine $A$'s for the quadratic vector

$$\beta_1 P_1 + \beta_2 P_2$$

which will always have at least one set of coplanar axes since it is a binomial. If then we can so choose our conditions that, according to the tests of Art. 9, there are three other sets of coplanar axes, the equations can be solved by quadratures.